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A Sharp Remez Inequality on the Size of Constrained Polynomials

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Denote by Π_n the set of all real algebraic polynomials of degree at most n. We define the class

$$\Pi_n(s) = \{p \in \Pi_n : m(\{x \in [-1, 1] : |p(x)| \leqslant 1\}) \geqslant 2 - s\} \qquad (0 < s < 2),$$

where $m(A)$ denotes the Lebesgue measure of A. How large can the maximum modulus be on $[-1, 1]$ for polynomials from $\Pi_n(s)$? In [7] E. J. Remez answered this question establishing the best possible upper bound. The solution and one of its applications in the theory of orthogonal polynomials can be found in [5] as well. Remez-type inequalities and their applications were studied in [1–3]. The purpose of this paper is to prove a sharp Remez-type inequality for constrained polynomials.

Remez's inequality asserts that

$$\max_{-1 \leqslant x \leqslant 1} |p(x)| \leqslant Q_n(4/(2-s) - 1) \qquad (p \in \Pi_n(s),\ 0 < s < 2), \tag{1}$$

where $Q_n(x) = \cos(n \arccos x)$ is the Chebyshev polynomial of degree n. For $a < b$ we define

$$P_n(a, b) = \left\{ p : p(x) = \sum_{j=0}^{n} \alpha_j (b-x)^j (x-a)^{n-j} \text{ with all } \alpha_j \geqslant 0 \text{ or all } \alpha_j \leqslant 0 \right\}.$$

The class $P_n(-1, 1)$ was introduced and examined thoroughly by G. G. Lorentz in [6], subsequently a number of properties were obtained in [4]. By an observation of Lorentz, if $p \in \Pi_n$ has no zero in the open unit circle then $p \in P_n(-1, 1)$. In this paper we prove the following sharp Remez-type theorem for polynomials from $P_n(-1, 1)$.

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THEOREM. *We have*

$$\max_{-1 \leqslant x \leqslant 1} |p(x)| \leqslant (1-s/2)^{-n} \qquad (p \in P_n(-1,1) \cap \Pi_n(s),\ 0<s<2), \quad (2)$$

and the equality holds only for the polynomials $\pm(1 \pm x)^n/(2-s)^n$.

COROLLARY. *If* $p \in \Pi_n(s)$ *has no zero in the open unit circle then* (2) *holds.*

Proof of the Theorem. Observe that $[c, d] \subset [a, b]$ implies $P_n(a, b) \subset P_n(c, d)$. This follows simply from the definition and the substitutions

$$b-x=\frac{b-c}{d-c}(d-x)+\frac{b-d}{d-c}(x-c),$$

$$x-a=\frac{c-a}{d-c}(d-x)+\frac{d-a}{d-c}(x-c),$$

where $(b-c)/(d-c)$, $(b-d)/(d-c)$, $(c-a)/(d-c)$, and $(d-a)/(d-c)$ are non-negative. Let $p \in P_n(a, b)$ with the representation

$$p(x)=\sum_{j=0}^{n} \alpha_j (b-x)^j (x-a)^{n-j} \qquad \text{with all } \alpha_j \geqslant 0 \text{ or all } \alpha_j \leqslant 0. \quad (3)$$

Then for $0<s<2$ we easily deduce

$$\begin{aligned} |p(b)| &= |\alpha_0|\,(b-a)^n=\left(\frac{b-a}{y-a}\right)^n |\alpha_0|\,(y-a)^n \leqslant \left(\frac{b-a}{y-a}\right)^n |p(y)| \\ &\leqslant (1-s/2)^{-n}\,|p(y)| \qquad (b-(b-a)\,s/2 \leqslant y \leqslant b) \end{aligned} \quad (4)$$

and similarly

$$|p(a)| \leqslant (1-s/2)^{-n}\,|p(y)| \qquad (a \leqslant y \leqslant a+(b-a)\,s/2). \quad (5)$$

Now let $p \in P_n(-1, 1) \cap \Pi_n(s)$ $(0<s<2)$, and choose a $z \in [-1, 1]$ such that

$$|p(z)| = \max_{-1 \leqslant x \leqslant 1} |p(x)|. \quad (6)$$

Since $p \in \Pi_n(s)$, there is a y from either $[z-s(z+1)/2, z]$ or $[z, z+s(1-z)/2]$ such that $|p(y)| \leqslant 1$. In the first case the relation $P_n(-1, 1) \subset P_n(-1, z)$ and (4) yield the desired result, and in the second case the relation $P_n(-1, 1) \subset P_n(z, 1)$ and (5) give the theorem.

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